Module
3
DC Transient

Lesson 11

Study of DC transients in R-L-C Circuits

Objectives

- Be able to write differential equation for a dc circuits containing two storage elements in presence of a resistance.
- To develop a thorough understanding how to find the complete solution of second order differential equation that arises from a simple $R-L-C$ circuit.
- To understand the meaning of the terms (i) overdamped (ii) criticallydamped, and (iii) underdamped in context with a second order dynamic system.
- Be able to understand some terminologies that are highly linked with the performance of a second order system.

L.11.1 Introduction

In the preceding lesson, our discussion focused extensively on dc circuits having resistances with either inductor (L) or capacitor (C) (i.e., single storage element) but not both. Dynamic response of such first order system has been studied and discussed in detail. The presence of resistance, inductance, and capacitance in the dc circuit introduces at least a second order differential equation or by two simultaneous coupled linear first order differential equations. We shall see in next section that the complexity of analysis of second order circuits increases significantly when compared with that encountered with first order circuits. Initial conditions for the circuit variables and their derivatives play an important role and this is very crucial to analyze a second order dynamic system.

L.11.2 Response of a series R-L-C circuit due to a dc voltage source

Consider a series $R-L-C$ circuit as shown in fig.11.1, and it is excited with a dc voltage source V_s. Applying KVL around the closed path for $t>0$,

$$L\frac{di(t)}{dt} + R\,i(t) + v_c(t) = V_s \qquad (11.1)$$

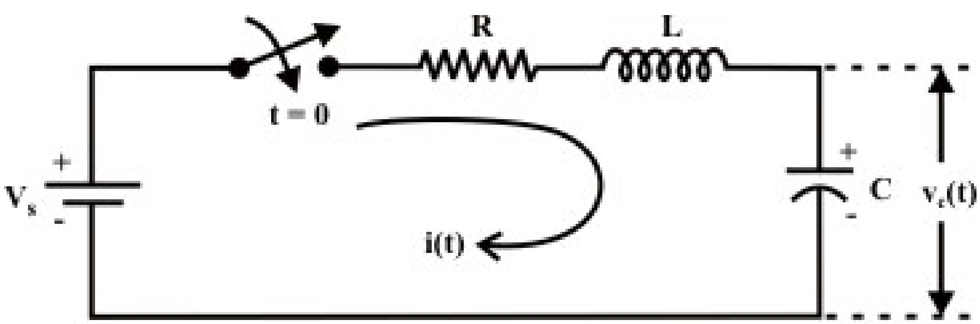

Fig. 11.1: A Simple R-L-C circuit excited with a dc voltage source

The current through the capacitor can be written as

$$i(t) = C\frac{dv_c(t)}{dt}$$

Substituting the current '$i(t)$'expression in eq.(11.1) and rearranging the terms,

$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = V_s \tag{11.2}$$

The above equation is a 2nd-order linear differential equation and the parameters associated with the differential equation are constant with time. The complete solution of the above differential equation has two components; the transient response $v_{cn}(t)$ and the steady state response $v_{cf}(t)$. Mathematically, one can write the complete solution as

$$v_c(t) = v_{cn}(t) + v_{cf}(t) = \left(A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}\right) + A \tag{11.3}$$

Since the system is linear, the nature of steady state response is same as that of forcing function (input voltage) and it is given by a constant value A. Now, the first part $v_{cn}(t)$ of the total response is completely dies out with time while $R > 0$ and it is defined as a transient or natural response of the system. The natural or transient response (see Appendix in Lesson-10) of second order differential equation can be obtained from the homogeneous equation (i.e., from force free system) that is expressed by

$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = 0 \Rightarrow \frac{d^2v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + \frac{1}{LC}v_c(t) = 0$$

$$a\,\frac{d^2v_c(t)}{dt^2} + b\,\frac{dv_c(t)}{dt} + c\,v_c(t) = 0 \;\; (\text{where } a = 1,\, b = \frac{R}{L} \textit{ and } c = \frac{1}{LC}) \tag{11.4}$$

The characteristic equation of the above homogeneous differential equation (using the operator $\alpha = \frac{d}{dt}$, $\alpha^2 = \frac{d^2}{dt^2}$ and $v_c(t) \neq 0$) is given by

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0 \quad \Rightarrow a\alpha^2 + b\alpha + c = 0 \;\; (\text{where } a = 1,\, b = \frac{R}{L} \textit{ and } c = \frac{1}{LC}) \tag{11.5}$$

and solving the roots of this equation (11.5) one can find the constants α_1 *and* α_2 of the exponential terms that associated with transient part of the complete solution (eq.11.3) and they are given below.

$$\alpha_1 = \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = \left(-\frac{b}{2a} + \frac{1}{a}\sqrt{\left(\frac{b}{2}\right)^2 - ac}\right);$$

$$\alpha_2 = \left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = \left(-\frac{b}{2a} - \frac{1}{a}\sqrt{\left(\frac{b}{2}\right)^2 - ac}\right) \tag{11.6}$$

where, $b = \frac{R}{L}$ *and* $c = \frac{1}{LC}$.

The roots of the characteristic equation (11.5) are classified in three groups depending upon the values of the parameters R, L, and C of the circuit.

Case-A (overdamped response): When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$, this implies that the roots are distinct with negative real parts. Under this situation, the natural or transient part of the complete solution is written as

$$v_{cn}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \tag{11.7}$$

and each term of the above expression decays exponentially and ultimately reduces to zero as $t \to \infty$ and it is termed as overdamped response of input free system. A system that is overdamped responds slowly to any change in excitation. It may be noted that the exponential term $A_1 e^{\alpha_1 t}$ takes longer time to decay its value to zero than the term $A_1 e^{\alpha_2 t}$. One can introduce a factor ξ that provides an information about the speed of system response and it is defined by damping ratio

$$(\xi) = \frac{Actual\ \ damping}{critical\ \ damping} = \frac{b}{2\sqrt{ac}} = \frac{R/L}{2/\sqrt{LC}} > 1 \tag{11.8}$$

Case-B (critically damped response): When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0$, this implies that the roots of eq.(11.5) are same with negative real parts. Under this situation, the form of the natural or transient part of the complete solution is written as

$$v_{cn}(t) = \left(A_1 t + A_2\right) e^{\alpha t} \qquad \text{(where } \alpha = -\frac{R}{2L}\text{)} \tag{11.9}$$

where the natural or transient response is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term. The expression (11.9) that arises from the natural solution of second order differential equation having the roots of characteristic equation are same value can be verified following the procedure given below.

The roots of this characteristic equation (11.5) are same $\alpha = \alpha_1 = \alpha_2 = \frac{R}{2L}$ when

$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0 \Rightarrow \left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$ and the corresponding homogeneous equation (11.4) can be rewritten as

$$\frac{d^2 v_c(t)}{dt^2} + 2\frac{R}{2L}\frac{dv_c(t)}{dt} + \frac{1}{LC} v_c(t) = 0$$

$$or\ \frac{d^2 v_c(t)}{dt^2} + 2\alpha\frac{dv_c(t)}{dt} + \alpha^2 v_c(t) = 0$$

$$or\ \frac{d}{dt}\left(\frac{dv_c(t)}{dt} + \alpha v_c(t)\right) + \alpha\left(\frac{dv_c(t)}{dt} + \alpha v_c(t)\right) = 0$$

$$or\ \frac{df}{dt} + \alpha f = 0 \qquad \text{where} \quad f = \frac{dv_c(t)}{dt} + \alpha v_c(t)$$

The solution of the above first order differential equation is well known and it is given by

$f = A_1\, e^{\alpha t}$

Using the value of f in the expression $f = \dfrac{dv_c(t)}{dt} + \alpha\, v_c(t)$ we can get,

$$\frac{dv_c(t)}{dt} + \alpha\, v_c(t) = A_1\, e^{-\alpha t} \Rightarrow e^{\alpha t}\frac{dv_c(t)}{dt} + e^{\alpha t}\, \alpha\, v_c(t) = A_1 \;\; \Rightarrow \frac{d}{dt}\left(e^{\alpha t}\, v_c(t)\right) = A_1$$

Integrating the above equation in both sides yields,

$v_{cn}(t) = \left(A_1 t + A_2\right) e^{\alpha t}$

In fact, the term $A_2\, e^{\alpha t}$ (with $\alpha = -\dfrac{R}{2L}$) decays exponentially with the time and tends to zero as $t \to \infty$. On the other hand, the value of the term $A_1 t\, e^{\alpha t}$ (with $\alpha = -\dfrac{R}{2L}$) in equation (11.9) first increases from its zero value to a maximum value $A_1 \dfrac{2L}{R} e^{-1}$ at a time $t = -\dfrac{1}{\alpha} = -\left(-\dfrac{2L}{R}\right) = \dfrac{2L}{R}$ and then decays with time, finally reaches to zero. One can easily verify above statements by adopting the concept of maximization problem of a single valued function. The second order system results the speediest response possible without any overshoot while the roots of characteristic equation (11.5) of system having the same negative real parts. The response of such a second order system is defined as a critically damped system's response. In this case damping ratio

$$(\xi) = \frac{Actual\;\; damping}{critical\;\; damping} = \frac{b}{2\sqrt{ac}} = \frac{R/L}{2/\sqrt{LC}} = 1 \qquad (11.10)$$

Case-C (underdamped response): When $\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC} < 0$, this implies that the roots of eq.(11.5) are complex conjugates and they are expressed as $\alpha_1 = \left(-\dfrac{R}{2L} + j\sqrt{\dfrac{1}{LC} - \left(\dfrac{R}{2L}\right)^2}\right) = \beta + j\gamma$; $\alpha_2 = \left(-\dfrac{R}{2L} - j\sqrt{\dfrac{1}{LC} - \left(\dfrac{R}{2L}\right)^2}\right) = \beta - j\gamma$. The form of the natural or transient part of the complete solution is written as

$$\begin{aligned} v_{cn}(t) &= A_1 e^{\alpha_1 t} + A_2\, e^{\alpha_2 t} = A_1 e^{(\beta + j\gamma)} + A_2\, e^{(\beta - j\gamma)} \\ &= e^{\beta t}\left[\left(A_1 + A_2\right)\cos(\gamma t) + j\left(A_1 - A_2\right)\sin(\gamma t)\right] \qquad (11.11) \\ &= e^{\beta t}\left[B_1 \cos(\gamma t) + B_2 \sin(\gamma t)\right] \text{ where } B_1 = A_1 + A_2\, ;\; B_2 = j\left(A_1 - A_2\right) \end{aligned}$$

For real system, the response $v_{cn}(t)$ must also be real. This is possible only if A_1 *and* A_2 conjugates. The equation (11.11) further can be simplified in the following form:

$$e^{\beta t}\, K \sin(\gamma t + \theta) \qquad (11.12)$$

where $\beta =$ real part of the root , $\gamma =$ complex part of the root, $K=\sqrt{B_1^{\,2}+B_2^{\,2}}$ *and* $\theta=\tan^{-1}\left(\dfrac{B_1}{B_2}\right)$. Truly speaking the value of K *and* θ can be calculated using the initial conditions of the circuit. The system response exhibits oscillation around the steady state value when the roots of characteristic equation are complex and results an under-damped system's response. This oscillation will die down with time if the roots are with negative real parts. In this case the damping ratio

$$(\xi)=\frac{Actual\ \ damping}{critical\ \ damping}=\frac{b}{2\sqrt{ac}}=\frac{R/L}{2/\sqrt{LC}}<1 \qquad (11.13)$$

Finally, the response of a second order system when excited with a dc voltage source is presented in fig.L.11.2 for different cases, i.e., (i) under-damped (ii) over-damped (iii) critically damped system response.

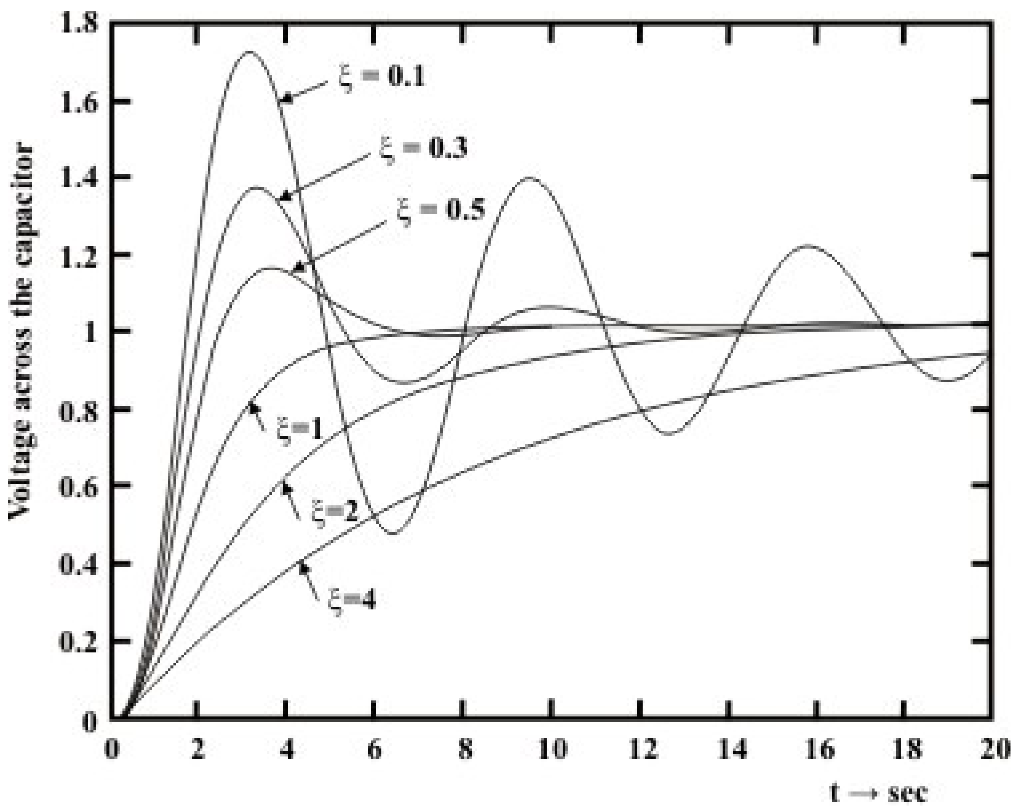

Fig. 11.2: System response for series R-L-C circuit:
(a) underdamped
(b) critically damped
(c) overdamped system

Example: L.11.1 The switch $S1$ was closed for a long time as shown in fig.11.3. Simultaneously at $t=0$, the switch $S1$ is opened and $S2$ is closed Find $(a)\ i_L(0^+)$; $(b)\ v_c(0^+)$; $(c)\ i_R(0^+)$; $(d)\ v_L(0^+)$; $(e)\ i_c(0^+)$; $(f)\ \frac{dv_c(0^+)}{dt}$.

Solution: When the switch $S1$ is kept in position '1' for a sufficiently long time, the circuit reaches to its steady state condition. At time $t=0^-$, the capacitor is completely charged and it acts as a open circuit. On other hand,

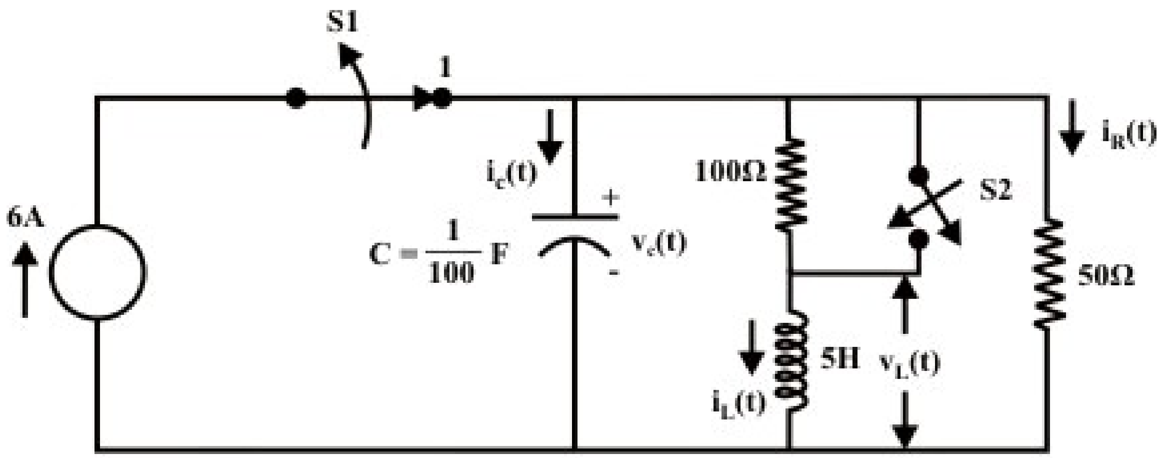

Fig. 11.3

the inductor acts as a short circuit under steady state condition, the current in inductor can be found as

$$i_L(0^-)=\frac{50}{100+50}\times 6=2\,A$$

Using the KCL, one can find the current through the resistor $i_R(0^-)=6-2=4A$ and subsequently the voltage across the capacitor $v_c(0^-)=4\times 50=200\,volt.$

Note at $t=0^+$ not only the current source is removed, but 100Ω resistor is shorted or removed as well. The continuity properties of inductor and capacitor do not permit the current through an inductor or the voltage across the capacitor to change instantaneously. Therefore, at $t=0^+$ the current in inductor, voltage across the capacitor, and the values of other variables at $t=0^+$ can be computed as

$i_L(0^+)=i_L(0^-)=2\,A;\quad v_c(0^+)=v_c(0^-)=200\,volt.$

Since the voltage across the capacitor at $t=0^+$ is $200\,volt$, the same voltage will appear across the inductor and the 50Ω resistor. That is, $v_L(0^+)=v_R(0^+)=200\,volt.$ and hence, the current $\left(i_R(0^+)\right)$ in 50Ω resistor = $\frac{200}{50}=4\,A$. Applying KCL at the bottom terminal of the capacitor we obtain $i_c(0^+) = -(4+2) = -6\,A$ and subsequently, $\frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{-6}{0.01} = -600\ volt./\sec.$

Example: L.11.2 The switch 'S' is closed sufficiently long time and then it is opened at time '$t=0$' as shown in fig.11.4. Determine $(i)\ v_0(0^+)\ (ii) \left.\frac{dv_c(t)}{dt}\right|_{t=0^+}\ (iii)\ i_L(0^+),\ and\ (iv) \left.\frac{di_L(t)}{dt}\right|_{t=0^+}\ (v) \left.\frac{dv_0(t)}{dt}\right|_{t=0^+}$ when $R_1 = R_2 = 3\Omega$.

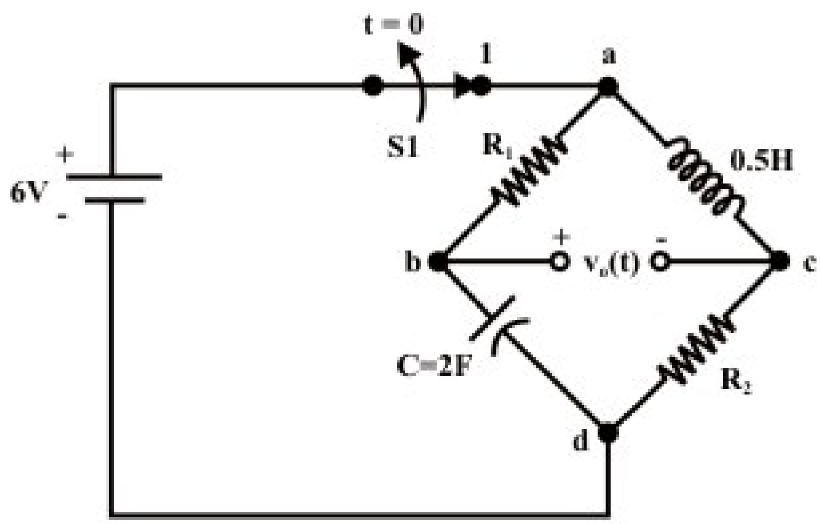

Fig. 11.4

Solution: At $t=0^-$ (just before opening the switch), the capacitor is fully charged and current flowing through it totally blocked i.e., capacitor acts as an open circuit). The voltage across the capacitor is $v_c(0^-) = 6V = v_c(0^+) = v_{bd}(0^+)$ and terminal 'b' is higher potential than terminal 'd'. On the other branch, the inductor acts as a short circuit (i.e., voltage across the inductor is zero) and the source voltage $6V$ will appear across the resistance R_2. Therefore, the current through inductor $i_L(0^-) = \frac{6}{3} = 2\,A = i_L(0^+)$. Note at $t=0^+$, $v_{ad}(0^+) = 0$ (since the voltage drop across the resistance $R_1 = 3\Omega = v_{ab} = -6V$) and $v_{cd}(0^+) = 6V$ and this implies that $v_{ca}(0^+) = 6V =$ voltage across the inductor (note, terminal 'c' is + ve terminal and inductor acts as a source of energy).

Now, the voltage across the terminals 'b' and 'c' ($v_0(0^+)) = v_{bd}(0^+) - v_{cd}(0^+) = 0V$.

The following expressions are valid at $t = 0^+$

$C\left.\frac{dv_c}{dt}\right|_{t=0^+} = i_c(0^+) = 2A \Rightarrow \left.\frac{dv_c}{dt}\right|_{t=0^+} = 1\ volt/\sec.$ (note, voltage across the capacitor will decrease with time i.e., $\left.\frac{dv_c}{dt}\right|_{t=0^+} = -1\ volt/\sec$). We have just calculated the voltage across the inductor at $t=0^+$ as

$$v_{ca}(0^+) = L\left.\frac{di_L(t)}{dt}\right|_{t=0^+} = 6V \;\Rightarrow\; \left.\frac{di_L(t)}{dt}\right|_{t=0^+} = \frac{6}{0.5} = 12\,A/\sec.$$

Now, $\frac{dv_0(0^+)}{dt} = \frac{dv_c(0^+)}{dt} - R_2\frac{di_L(0^+)}{dt} = 1 - (12\times 3) = -35\,volt/\sec.$

Example: L.11.3 Refer to the circuit in fig.11.5(a). Determine,

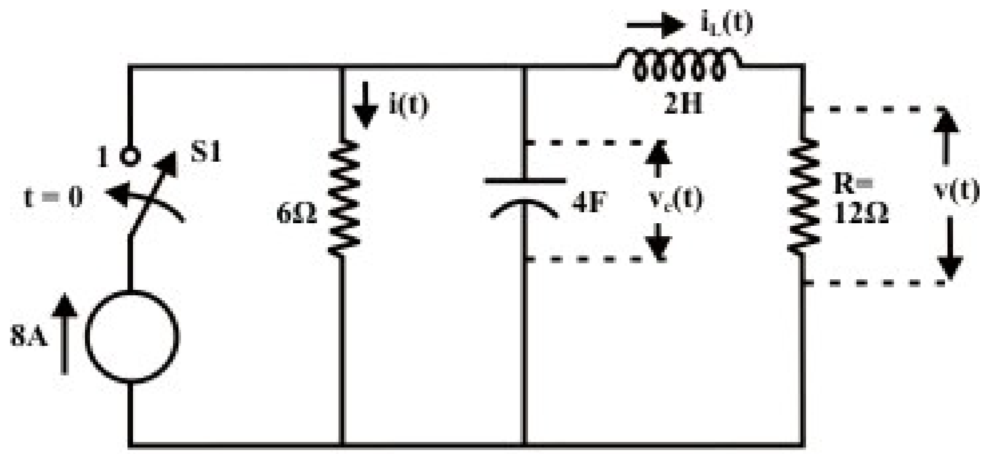

Fig. 11.5(a)

(i) $i(0^+), i_L(0^+)$ *and* $v(0^+)$ (ii) $\frac{di(0^+)}{dt}$ *and* $\frac{dv(0^+)}{dt}$ **(iii)** $i(\infty),\ i_L(\infty)$ *and* $v(\infty)$ (assumed $v_c(0)=0$; $i_L(0)=0$)

Solution: When the switch was in 'off' position i.e., t < 0

$$i(0^-) = i_L(0^-) = 0,\ v(0^-) = 0 \text{ and } v_C(0^-) = 0$$

The switch '$S1$' was closed in position '1' at time t = 0 and the corresponding circuit is shown in fig 11.5 (b).

(i) From continuity property of inductor and capacitor, we can write the following expression for t = 0^+

$$i_L(0^+) = i_L(0^-) = 0,\ v_c(0^+) = v_c(0^-) = 0 \;\Rightarrow\; i(0^+) = \frac{1}{6}v_c(0^+) = 0$$

$$v(0^+) = i_L(0^+)\times 6 = 0 \text{ volt}.$$

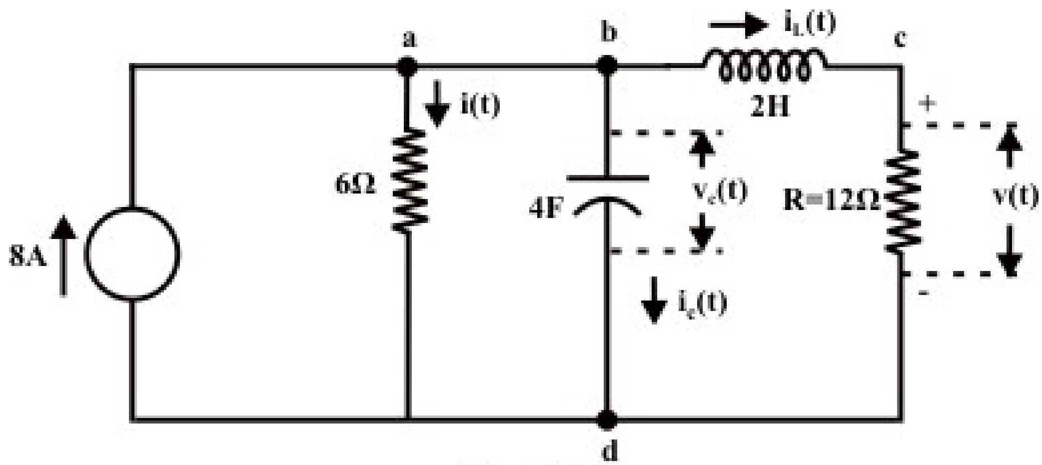

Fig. 11.5(b)

(ii) KCL at point 'a'

$$8 = i(t) + i_c(t) + i_L(t)$$

At $t = 0^+$, the above expression is written as

$$8 = i(0^+) + i_c(0^+) + i_L(0^+) \qquad \Rightarrow i_c(0^+) = 8\,A$$

We know the current through the capacitor $i_c(t)$ can be expressed as

$$i_c(t) = C\frac{dv_c(t)}{dt}$$

$$i_c(0^+) = C\frac{dv_c(0^+)}{dt}$$

$$\therefore \quad \frac{dv_c(0^+)}{dt} = 8 \times \frac{1}{4} = 2 \text{ volt./sec.}.$$

Note the relations

$\frac{dv_c(0^+)}{dt}$ = change in voltage drop in 6Ω resistor = change in current through 6Ω

resistor$\times 6 = 6\times\frac{di(0^+)}{dt} \qquad \Rightarrow \frac{di(0^+)}{dt} = \frac{2}{6} = \frac{1}{3}$ $amp./\sec.$

Applying KVL around the closed path 'b-c-d-b', we get the following expression.

$v_c(t) = v_L(t) + v(t)$

At, $t = 0^+$ the following expression

$$v_c(0^+) = v_L(0^+) + i_L(0^+)\times 12$$

$$0 = v_L(0^+) + 0\times 12 \Rightarrow v_L(0^+) = 0 \Rightarrow L\frac{di_L(0^+)}{dt} = 0 \Rightarrow \frac{di_L(0^+)}{dt} = 0$$

$$\frac{di_L(0^+)}{dt} = 0 \text{ and this implies } 12\frac{di_L(0^+)}{dt} = 12\times 0 = 0 \text{ v/sec} = \frac{dv(0^+)}{dt} = 0$$

Now, $v(t) = R\,i_L(t)$ *also* at $t = 0^+$

$$\frac{dv(0^+)}{dt} = R\frac{di_L(0^+)}{dt} = 12\,\frac{di_L(0^+)}{dt} = 0\ volt/\text{sec}.$$

(iii) At $t = \alpha$, the circuit reached its steady state value, the capacitor will block the flow of dc current and the inductor will act as a short circuit. The current through $6\,\Omega$ and 12 Ω resistors can be formed as

$$\text{i}(\infty) = \frac{12\times 8}{18} = \frac{16}{3} = 5.333\text{A},\ \ \text{i}_\text{L}(\infty) = 8 - 5.333 = 2.667\,\text{A}$$

$$v_c(\infty) = 32\ volt.$$

Example: L.11.4 The switch $S1$ has been closed for a sufficiently long time and then it is opened at $t=0$ (see fig.11.6(a)). Find the expression for (a) $v_c(t)$, (b) $i_c(t)$, $t>0$ for inductor values of $(i)\,L=0.5\,H$ $(ii)\ L=0.2\,H$ $(iii)\ L=1.0\,H$ and plot $v_c(t)-vs-t$ and $i(t)-vs-t$ for each case.

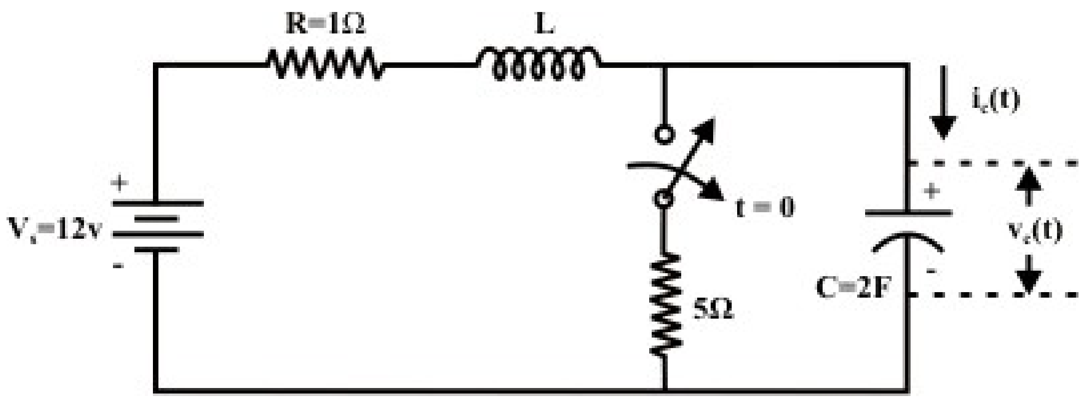

Fig. 11.6(a)

Solution: At $t=0^-$ (before the switch is opened) the capacitor acts as an open circuit or block the current through it but the inductor acts as short circuit. Using the properties of inductor and capacitor, one can find the current in inductor at time $t = 0^+$ as

$i_L(0^+) = i_L(0^-) = \dfrac{12}{1+5} = 2A$ (note inductor acts as a short circuit) and voltage across the 5Ω resistor = $2\times 5 = 10\,volt$. The capacitor is fully charged with the voltage across the 5Ω resistor and the capacitor voltage at $t = 0^+$ is given by

$v_c(0^+) = v_c(0^-) = 10\,volt.$ The circuit is opened at time $t=0$ and the corresponding circuit diagram is shown in fig. 11.6(b).

Case-1: $L=0.5H$, $R=1\Omega$ *and* $C=2F$

Let us assume the current flowing through the circuit is $i(t)$ and apply KVL equation around the closed path is

$$V_s = R\,i(t) + L\frac{di(t)}{dt} + v_c(t) \Rightarrow V_s = RC\frac{dv_c(t)}{dt} + LC\frac{d^2v_c(t)}{dt^2} + v_c(t) \text{ (note, } i(t) = C\frac{dv_c(t)}{dt}\text{)}$$

$$V_s = \frac{d^2v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + + \frac{1}{LC}v_c(t) \qquad (11.14)$$

The solution of the above differential equation is given by

$$v_c(t) = v_{cn}(t) + v_{cf}(t) \qquad (11.15)$$

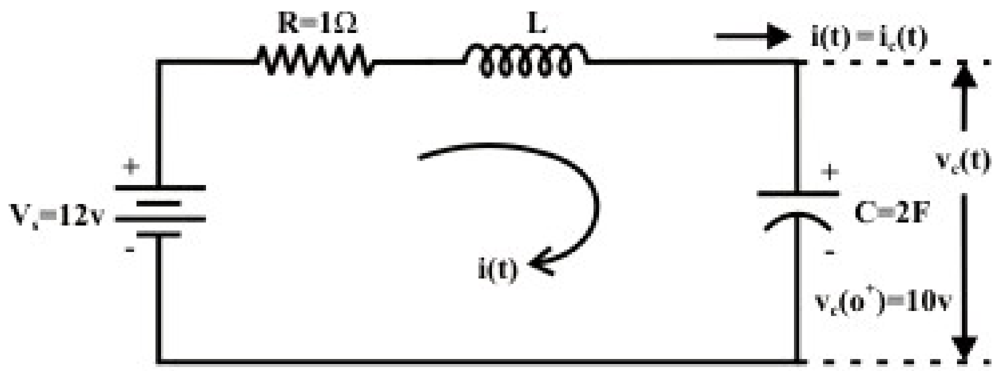

Fig. 11.6(b)

The solution of natural or transient response $v_{cn}(t)$ is obtained from the force free equation or homogeneous equation which is

$$\frac{d^2v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + \frac{1}{LC}v_c(t) = 0 \qquad (11.16)$$

The characteristic equation of the above homogeneous equation is written as

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0 \qquad (11.17)$$

The roots of the characteristic equation are given as

$$\alpha_1 = \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = -1.0; \quad \alpha_2 = \left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) = -1.0$$

and the roots are equal with negative real sign. The expression for natural response is given by

$$v_{cn}(t) = \left(A_1 t + A_2\right)e^{\alpha t} \quad (\text{where } \alpha = \alpha_1 = \alpha_2 = -1) \qquad (11.18)$$

The forced or the steady state response $v_{cf}(t)$ is the form of applied input voltage and it is constant 'A'. Now the final expression for $v_c(t)$ is

$$v_c(t) = \left(A_1 t + A_2\right)e^{\alpha t} + A = \left(A_1 t + A_2\right)e^{-t} + A \qquad (11.19)$$

The initial and final conditions needed to evaluate the constants are based on $v_c(0^+) = v_c(0^-) = 10\,volt$; $i_L(0^+) = i_L(0^-) = 2\,A$ (Continuity property).

At $t=0^+$;

$$v_c(t)\big|_{t=0^+} = A_2\, e^{-1\times 0} + A = A_2 + A \quad \Rightarrow \; A_2 + A = 10 \tag{11.20}$$

Forming $\dfrac{dv_c(t)}{dt}$ (from eq.(11.19)as

$$\frac{dv_c(t)}{dt} = \alpha\left(A_1 t + A_2\right)e^{\alpha t} + A_1\, e^{\alpha t} = -\left(A_1 t + A_2\right)e^{-t} + A_1\, e^{-t}$$

$$\left.\frac{dv_c(t)}{dt}\right|_{t=0^+} = A_1 - A_2 \Rightarrow A_1 - A_2 = 1 \tag{11.21}$$

(note, $C\dfrac{dv_c(0^+)}{dt} = i_c(0^+) = i_L(0^+) = 2 \Rightarrow \dfrac{dv_c(0^+)}{dt} = 1\, volt/\sec.$)

It may be seen that the capacitor is fully charged with the applied voltage when $t=\infty$ and the capacitor blocks the current flowing through it. Using $t=\infty$ in equation (11.19) we get,

$v_c(\infty) = A \Rightarrow A = 12$

Using the value of A in equation (11.20) and then solving (11.20) and (11.21) we get, $A_1 = -1;\ A_2 = -2$.

The total solution is

$$\begin{aligned} v_c(t) &= -\left(t+2\right)e^{-t} + 12 = 12 - \left(t+2\right)e^{-t}; \\ i(t) &= C\frac{dv_c(t)}{dt} = 2\times\left[\left(t+2\right)e^{-t} - e^{-t}\right] = 2\times\left(t+1\right)e^{-t} \end{aligned} \tag{11.22}$$

The circuit responses (critically damped) for $L = 0.5\,H$ are shown fig.11.6 (c) and fig.11.6(d).

Case-2: $L = 0.2\,H, R = 1\Omega \; and \; C = 2F$

It can be noted that the initial and final conditions of the circuit are all same as in case-1 but the transient or natural response will differ. In this case the roots of characteristic equation are computed using equation (11.17), the values of roots are

$\alpha_1 = -0.563;\ \alpha_2 = -4.436$

The total response becomes

$$v_c(t) = A_1 e^{\alpha_1 t} + A_2\, e^{\alpha_2 t} + A = A_1 e^{-4.436t} + A_2\, e^{-0.563t} + A \tag{11.23}$$

$$\frac{dv_c(t)}{dt} = \alpha_1\, A_1\, e^{\alpha_1 t} + \alpha_2 A_2\, e^{\alpha_2 t} = -4.435\, A_1\, e^{-4.436t} - 0.563\, A_2\, e^{-0.536t} \tag{11.24}$$

Using the initial conditions($v_c(0^+) = 10$, $\dfrac{dv_c(0^+)}{dt} = 1\, volt/\sec.$) that obtained in case-1 are used in equations (11.23)-(11.24) with $A = 12$ (final steady state condition) and simultaneous solution gives

$A_1 = 0.032;\ A_2 = -2.032$

The total response is

$$v_c(t) = 0.032\, e^{-4.436t} - 2.032\, e^{-0.563t} + 12$$

$$i(t) = C\frac{dv_c(t)}{dt} = 2\left[1.14\, e^{-0.563t} - 0.14\, e^{-4.436t}\right] \qquad (11.25)$$

The system responses (overdamped) for $L = 0.2\,H$ are presented in fig.11.6(c) and fig.11.6 (d).

Case-3: $L = 8.0\,H, R = 1\Omega \; and \; C = 2F$

Again the initial and final conditions will remain same and the natural response of the circuit will be decided by the roots of the characteristic equation and they are obtained from (11.17) as

$\alpha_1 = \beta + j\gamma = -0.063 + j\,0.243;\; \alpha_2 = \beta - j\gamma = -0.063 - j\,0.242$

The expression for the total response is

$$v_c(t) = v_{cn}(t) + v_{cf}(t) = e^{\beta t}\; K\sin\left(\gamma t + \theta\right) + A \qquad (11.26)$$

(note, the natural response $v_{cn}(t) = e^{\beta t}\; K\sin\left(\gamma t + \theta\right)$ is written from eq.(11.12) when roots are complex conjugates and detail derivation is given there.)

$$\frac{dv_c(t)}{dt} = K\, e^{\beta t}\left[\beta\sin\left(\gamma t + \theta\right) + \gamma\cos\left(\gamma t + \theta\right)\right] \qquad (11.27)$$

Again the initial conditions ($v_c(0^+) = 10$, $\frac{dv_c(0^+)}{dt} = 1\,volt/\sec.$) that obtained in case-1 are used in equations (11.26)-(11.27) with $A = 12$ (final steady state condition) and simultaneous solution gives

$K = 4.13;\; \theta = -28.98^0\; \left(\deg ree\right)$

The total response is

$$v_c(t) = e^{\beta t}\; K\sin\left(\gamma t + \theta\right) + 12 = e^{-0.063t}\; 4.13\sin\left(0.242t - 28.99^0\right) + 12$$

$$v_c(t) = 12 + 4.13\, e^{-0.063t}\; \sin\left(0.242t - 28.99^0\right) \qquad (11.28)$$

$$i(t) = C\frac{dv_c(t)}{dt} = 2e^{-0.063t}\left[0.999 * \cos\left(0.242t - 28.99^0\right) - 0.26\sin\left(0.242t - 28.99^0\right)\right]$$

The system responses (under-damped) for $L = 8.0\,H$ are presented in fig.11.6(c) and fig. 11.6(d).

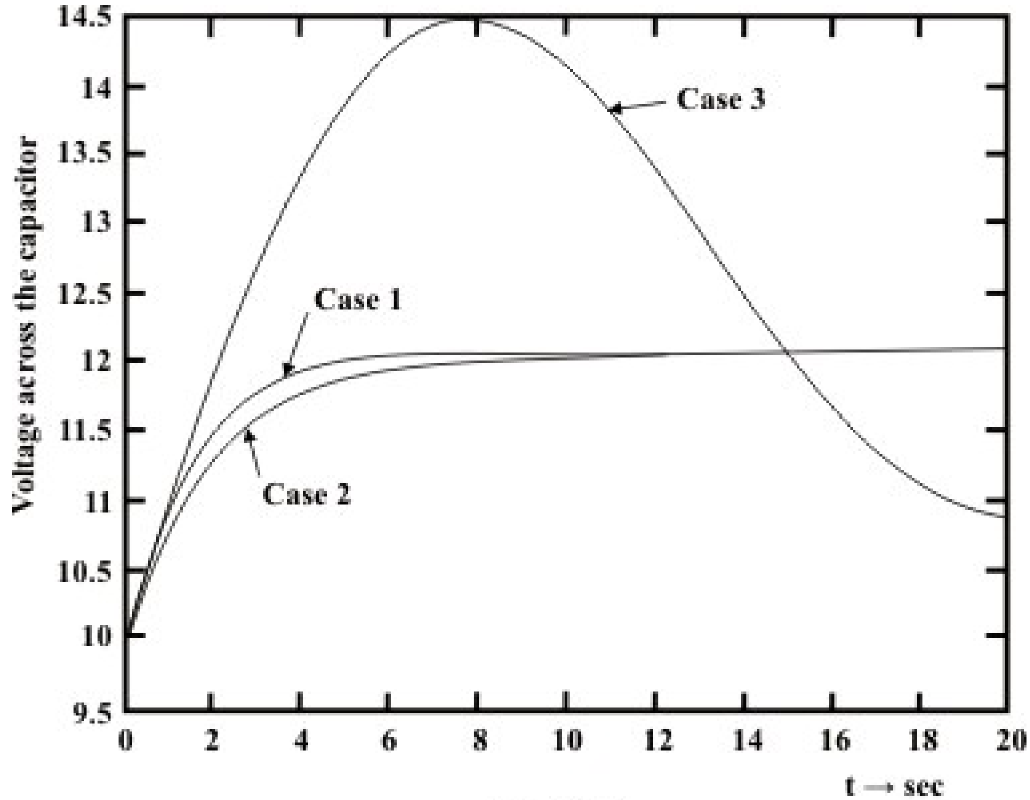

Fig. 11.6(c)

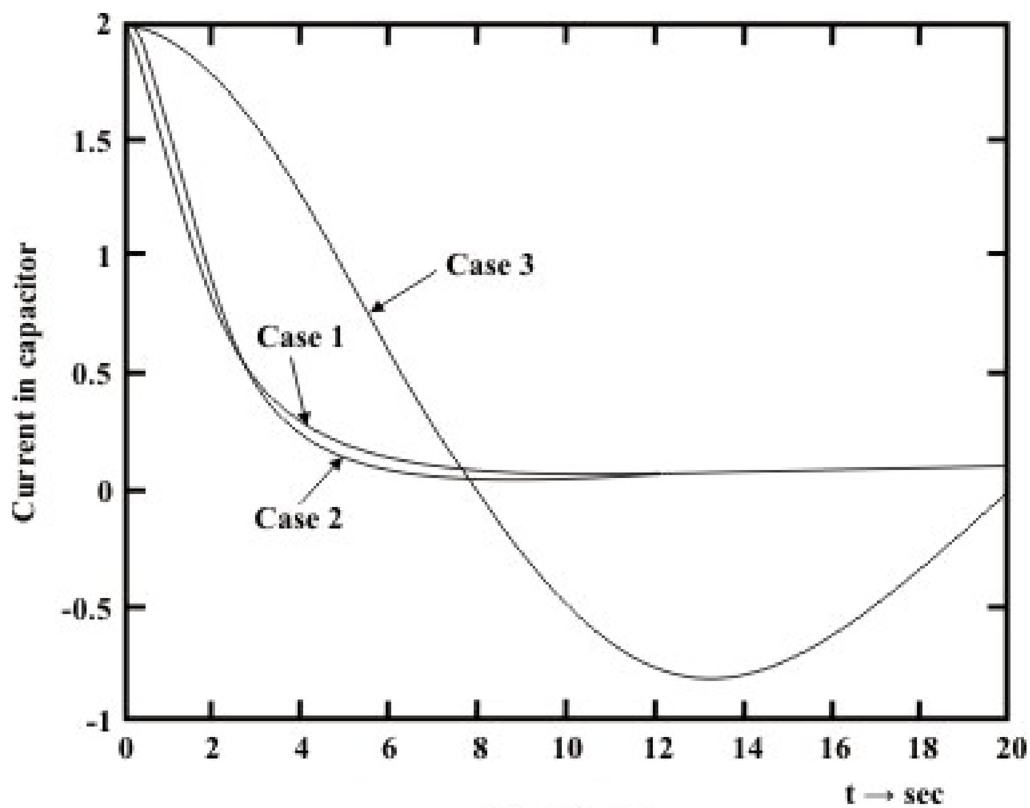

Fig. 11.6(d)

Remark: One can use $t = 0$ *and* $t = \infty$ in eq. 11.22 or eq. 11.25 or eq. 11.28 to verify whether it satisfies the initial and final conditions (i.e., initial capacitor voltage $v_c(0^+) = 10\,volt.$, and the steady state capacitor voltage $v_c(\infty) = 12\,volt.$) of the circuit.

Example: L.11.5 The switch '$S1$' in the circuit of Fig. 11.7(a) was closed in position '1' sufficiently long time and then kept in position '2'. Find (i) $v_c(t)$ (ii) $i_c(t)$ for t ≥ 0 if C is (a) $\frac{1}{9}F$ (b) $\frac{1}{4}F$ (c) $\frac{1}{8}F$.

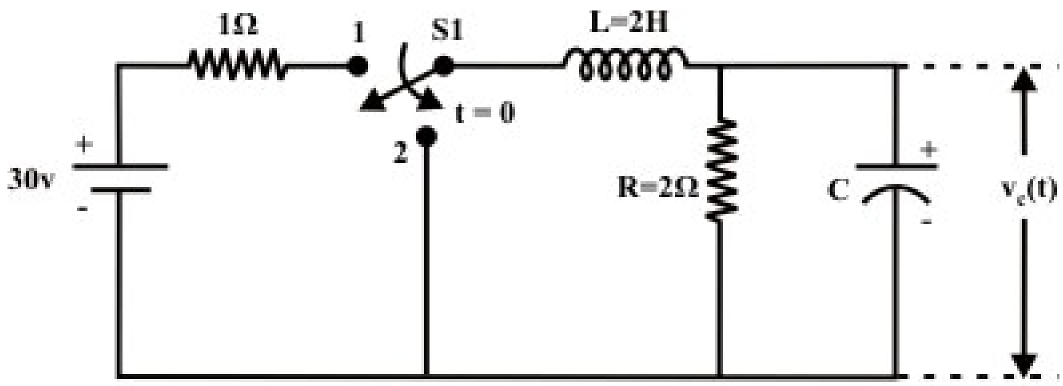

Fig. 11.7(a)

Solution: When the switch was in position '1', the steady state current in inductor is given by

$$i_L(0^-) = \frac{30}{1+2} = 10\text{A}, \quad v_c(0^-) = i_L(0^-)R = 10\times 2 = 20 \text{ volt.}$$

Using the continuity property of inductor and capacitor we get

$$i_L(0^+) = i_L(0^-) = 10, \quad v_c(0^+) = v_c(0^-) = 20 \text{ volt.}$$

The switch '$S1$' is kept in position '2' and corresponding circuit diagram is shown in Fig.11.7 (b)

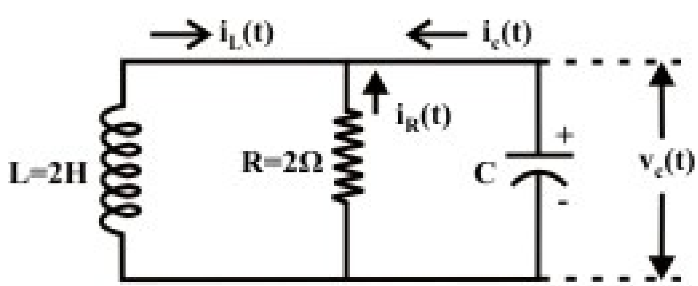

Fig. 11.7(b)

Applying KCL at the top junction point we get,

$$\frac{v_C(t)}{R} + i_C(t) + i_L(t) = 0$$

$$\frac{v_c(t)}{R} + C\frac{dv_c(t)}{dt} + i_L(t) = 0$$

$$\frac{L}{R}\frac{di_L(t)}{dt} + C.L\frac{d^2 i_L(t)}{dt^2} + i_L(t) = 0 \quad [\text{note: } v_c(t) = L\frac{di_L(t)}{dt}]$$

or
$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t) = 0 \qquad (11.29)$$

The roots of the characteristics equation of the above homogeneous equation can obtained for $C = \frac{1}{9}\ F$

$$\alpha_1 = \frac{-\frac{1}{RC} + \sqrt{\left(\frac{1}{RC}\right)^2 - 4/LC}}{2} = \frac{-\frac{9}{2} + \sqrt{\left(\frac{9}{2}\right)^2 - \frac{4\times 9}{2}}}{2} = -1.5$$

$$\alpha_2 = \frac{-\frac{1}{RC} - \sqrt{\left(\frac{1}{RC}\right)^2 - 4/LC}}{2} = \frac{-\frac{9}{2} - \sqrt{\left(\frac{9}{2}\right)^2 - \frac{4\times 9}{2}}}{2} = -3.0$$

Case-1 $\left(\xi = 1.06,\ \textit{over damped system}\right)$**:** $C = \frac{1}{9}F$, the values of roots of characteristic equation are given as

$\alpha_1 = -1.5$, $\alpha_2 = -3.0$

The transient or neutral solution of the homogeneous equation is given by

$$i_L(t) = A_1 e^{-1.5t} + A_2 e^{-3.0t} \qquad (11.30)$$

To determine A_1 and A_2, the following initial conditions are used.

At $t = 0^+$;

$$i_L(0^+) = i_L(0^-) = A_1 + A_2$$
$$10 = A_1 + A_2 \qquad (11.31)$$

$$v_c(0^+) = v_c(0^-) = v_L(0^+) = L\left.\frac{di_L(t)}{dt}\right|_{t=0^+}$$

$$20 = 2\times\left[A_1 \times -1.5\, e^{-1.5t} - 3.0 \times A_2 e^{-3.0t}\right] \qquad (11.32)$$
$$= 2\left[-1.5A_1 - 3A_2\right] = -3A_1 - 6A_2$$

Solving equations (11.31) and (11,32) we get , $A_2 = -16.66$, $A_1 = 26.666$.

The natural response of the circuit is

$$i_L = \frac{80}{3}e^{-1.5t} - \frac{50}{3}e^{-3.0t} = 26.66e^{-1.5t} - 16.66e^{-3.0t}$$

$$L\frac{di_L}{dt} = 2\left[26.66\times -1.5e^{-1.5t} - 16.66\times -3.0\,e^{-3.0t}\right]$$

$$v_L(t) = v_c(t) = \left[100\,e^{-3.0t} - 80\,e^{-1.5t}\right]$$

$$i_c(t) = c\frac{dv_c(t)}{dt} = \frac{1}{9}\left(-300.0\,e^{-3.0t} + 120\,e^{-1.5t}\right) = \left(13.33\,e^{-1.5t} - 33.33\,e^{-3.0t}\right)$$

Case-2 $(\xi = 0.707, under\ \ damped\ \ system)$**:** For $C = \frac{1}{4}F$, the roots of the characteristic equation are

$$\alpha_1 = -1.0 + j1.0 = \beta + j\gamma$$

$$\alpha_2 = -1.0 - j1.0 = \beta - j\gamma$$

The natural response becomes 1

$$i_L(t) = k\,e^{\beta t}\sin(\gamma t + \theta) \qquad (11.33)$$

Where k and θ are the constants to be evaluated from initial condition.

At $t = 0^+$, from the expression (11.33) we get,

$$i_L\left(0^+\right) = k\sin\theta$$

$$10 = k\sin\theta \qquad (11.34)$$

$$L\frac{di(t)}{dt}\bigg|_{t=0^+} = 2\times k\left[\beta\,e^{\beta t}\sin(\gamma t + \theta) + e^{\beta t}\gamma\cos(\gamma t + \theta)\right]\bigg|_{t=0^+} \qquad (11.35)$$

Using equation (11.34) and the values of $\beta\ and\ \gamma$ in equation (11.35) we get,

$$20 = 2k(\beta sn\theta + \gamma\cos\theta) = k\cos\,\theta \quad (\text{note: } \beta = -1,\ \gamma = 1\ \ and\ \ k\sin\theta = 10) \qquad (11.36)$$

From equation (11.34) and (11.36) we obtain the values of θ and k as

$$\tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.56^o \text{ and } k = \frac{10}{\sin\theta} = 22.36$$

∴ The natural or transient solution is

$$i_L(t) = 22.36\,e^{-t}\sin\left(t + 26.56^o\right)$$

$$L\frac{di(t)}{dt} = v_c(t) = 2\times k\times\left[\beta\sin(\gamma t + \theta) + \gamma\cos(\gamma t + \theta)\right]e^{\beta t}$$

$$= 44.72\left[\cos(t + 26.56^o) - \sin(t + 26.56^o)\right]\times e^{-t}$$

$$i_c(t) = c\frac{dv_c(t)}{dt} = \frac{1}{4}\times 44.72\frac{d}{dt}\left\{\left[\cos(t + 26.56^o) - \sin(t + 26.56^o)\right]e^{-t}\right.$$

$$= -22.36\cos(t + 26.56)\,e^{-t}$$

Case-3 $\left(\xi=1, critically\ damped\ system\right)$**:** For $C=\frac{1}{8}F$; the roots of characteristic equation are $\alpha_1=-2;\ \alpha_2=-2$ respectively. The natural solution is given by

$$i_L(t)=\left(A_1 t+A_2\right)e^{\alpha t} \tag{11.37}$$

where constants are computed using initial conditions.

At $t=0^+$; from equation (11.37) one can write

$$\mathrm{i_L}(0^+)=A_2 \Rightarrow A_2=10$$

$$\mathrm{L}\left.\frac{\mathrm{di(t)}}{\mathrm{dt}}\right|_{\mathrm{t}=0^+}=2\times\left[A_2\,\alpha\,e^{\alpha t}+\alpha A_1\,t\,e^{\alpha t}+A_1\,e^{\alpha t}\right]_{t=0^+}$$

$$=2\times\left[\left(A_1+A_2\alpha\right)e^{\alpha t}+\alpha A_1\,t\,e^{\alpha t}\right]_{t=0^+}$$

$$\mathrm{L}\left.\frac{\mathrm{di(t)}}{\mathrm{dt}}\right|_{\mathrm{t}=0^+}=v_c(0^+)=20=2\left(A_1-2A_2\right) \Rightarrow A_1=30$$

The natural response is then

$$i_L(t)=\left(10+30t\right)e^{-2t}$$

$$\mathrm{L}\frac{\mathrm{di_L(t)}}{\mathrm{dt}}=2\times\frac{d}{dt}\left[\left(10+30t\right)e^{-2t}\right]$$

$$\mathrm{L}\frac{\mathrm{di_L(t)}}{\mathrm{dt}}=v_c(t)=2\left[10-60t\right]e^{-2t}$$

$$i_c(t)=c\frac{dv_c(t)}{dt}=\frac{1}{8}\times2\times\frac{d}{dt}\left[\left(10-60t\right)e^{-2t}\right]=\left[-20e^{-2t}+30t\,e^{-2t}\right]$$

Case-4 $\left(\xi=2, over\ damped\ system\right)$: For $C=\frac{1}{32}F$

Following the procedure as given in case-1 one can obtain the expressions for (i) current in inductor $i_L(t)$ (ii) voltage across the capacitor $v_c(t)$

$$i_L(t)=11.5e^{-1.08t}-1.5e^{-14.93t}$$

$$L\frac{di(t)}{dt}=v_c(t)=\left[44.8\,e^{-14.93t}-24.8\,e^{-1.08t}\right]$$

$$i_c(t)=c\frac{dv_c(t)}{dt}=\frac{1}{32}\times\frac{d}{dt}\left[44.8\,e^{-14.93t}-24.8\,e^{-1.08t}\right]$$

$$=0.837e^{-1.08t}-20.902e^{-14.93t}$$

L.11.3 Test your understanding (Marks: 80)

T.11.1 Transient response of a second-order ----------------- dc network is the sum of two real exponentials. [1]

T.11.2 The complete response of a second order network excited from dc sources is the sum of -------- response and ---------------- response. [2]

T.11.3 Circuits containing two different classes of energy storage elements can be described by a ------------------- order differential equations. [1]

T.11.4 For the circuit in fig.11.8, find the following [6]

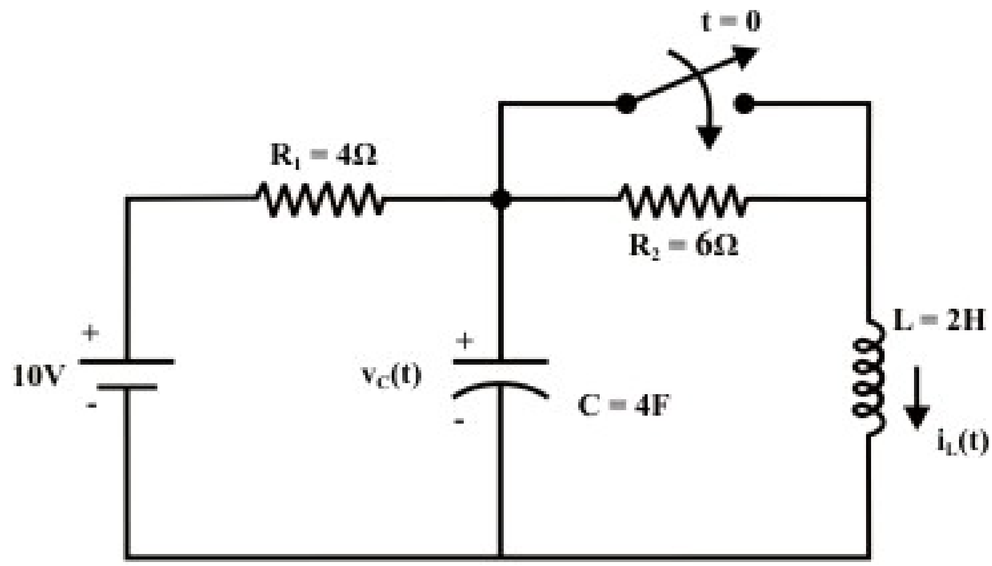

Fig. 11.8

$(a)\ v_c(0^-)\ \ (b)\ v_c(0^+)\ \ (c)\ \dfrac{dv_c(0^-)}{dt}\ \ (d)\ \dfrac{dv_c(0^+)}{dt}\ \ (e)\dfrac{di_L(0^-)}{dt}\ \ (f)\ \dfrac{di_L(0^+)}{dt}$

(Ans. (a) 6 $volt$. (b) 6 $volt$. (c) 0 V/sec. (d) 0 V/sec. (e) 0 amp/sec. (f) 3 $amp.$/sec.)

T.11.5 In the circuit of Fig. 11.9,

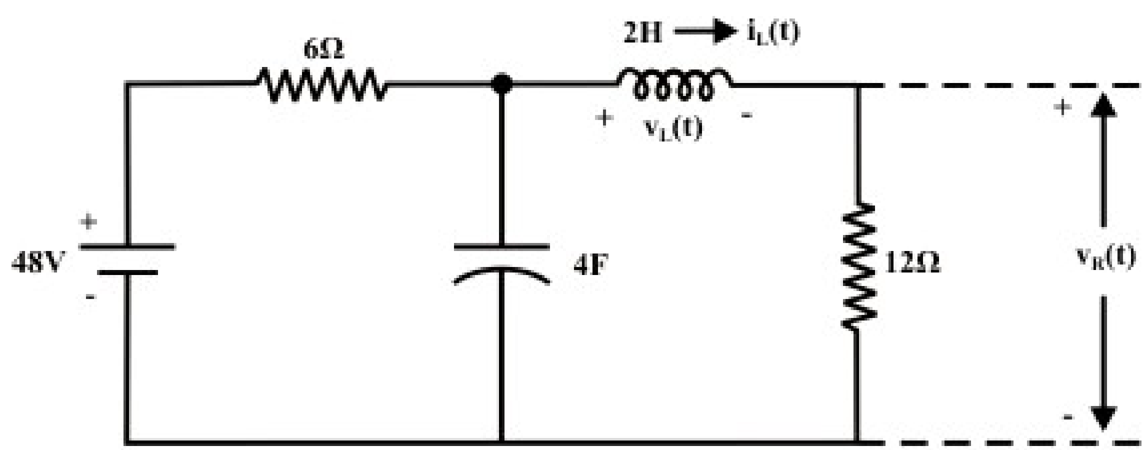

Fig. 11.9

Find,

$(a)\ v_R(0^+)\ and\ v_L(0^+)\ (b)\ \frac{dv_R(0^+)}{dt}\ and\ \frac{dv_L(0^+)}{dt}\ (c)\ v_R(\infty)\ and\ v_L(\infty)$ [8]

(Assume the capacitor is initially uncharged and current through inductor is zero).
(Ans. $(a)\ 0V,\ 0V\ \ (b)\ 0V,\ \ 2Volt./Sec.\ (c) 32V,\ 0V$)

T.11.6 For the circuit shown in fig.11.10, the expression for current through inductor

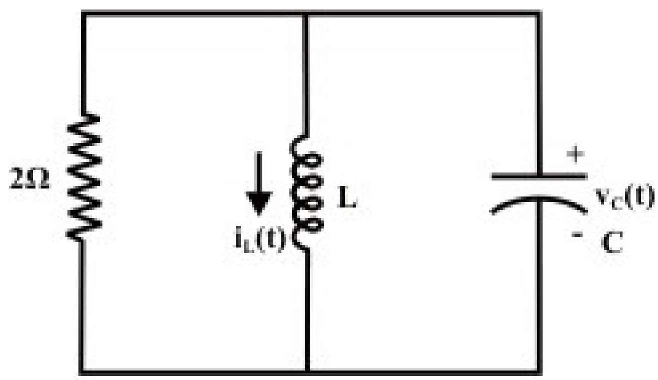

Fig. 11.10

is given by $i_L(t) = \left(10+30t\right)e^{-2t}\quad for\quad t \geq 0$

Find, (a) the values of L, C (b) initial condition $v_c(0^-)$ (c) the expression for $v_c(t) > 0$.

(Ans. $(a)\ L = 2H$, $C = \frac{1}{8}F$ $(b)\ v_c(0^-) = 20V$ $(c)\ v_c(t) = \left(20 - 120t\right)e^{-2t}\,V.$) [8]

T.11.7 The response of a series RLC circuit are given by

$$v_c(t) = 12 + 0.032\,e^{-4.436t} - 2.032\,e^{-0.563t}$$
$$i_L(t) = 2.28\,e^{-0.563t} - 0.28\,e^{-4.436t}$$

where $v_c(t)\ and\ i_L(t)$ are capacitor voltage and inductor current respectively. Determine (a) the supply voltage (b) the values $R,\ L,\ C$ of the series circuit. [4+4]

(Ans. $(a)\ 12V\ (b)\ R = 1\Omega, L = 0.2\,H\ and\ C = 2\,F$)

T.11.8 For the circuit shown in Fig. 11.11, the switch 'S'was in position '1' for a long time and then at $t = 0$ it is kept in position '2'.

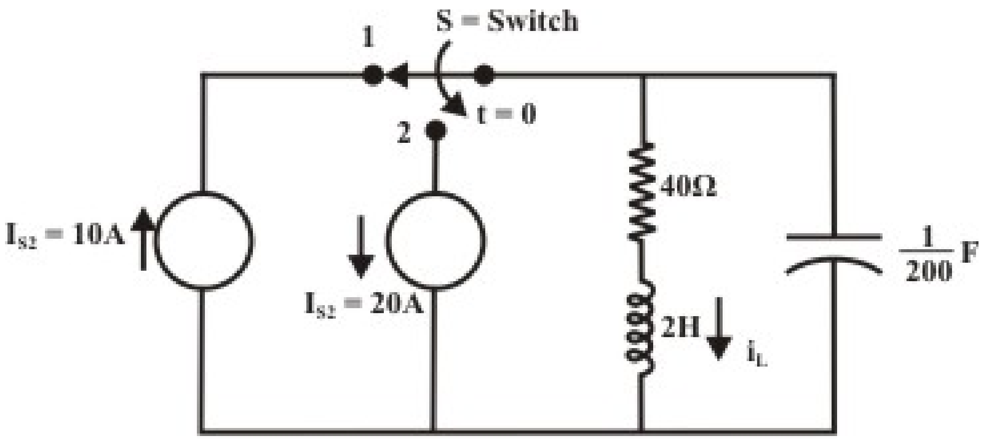

Fig. 11.11

Find,

$(a)\ i_L(0^-);\ (b)\ v_c(0^+);\ (c)\ v_R(0^+);\ (d)\ i_L(\infty);$ [8]

Ans.

$(a)\ i_L(0^-) = 10\,A\ ;\ (b)\ v_c(0^+) = 400V;$

$(c)\ v_R(0^+) = 400V \quad (d)\ i_L(\infty) = -20\,A$

T.11.9 For the circuit shown in Fig.11.12, the switch 'S' has been in position '1' for a long time and at $t = 0$ it is instantaneously moved to position '2'.

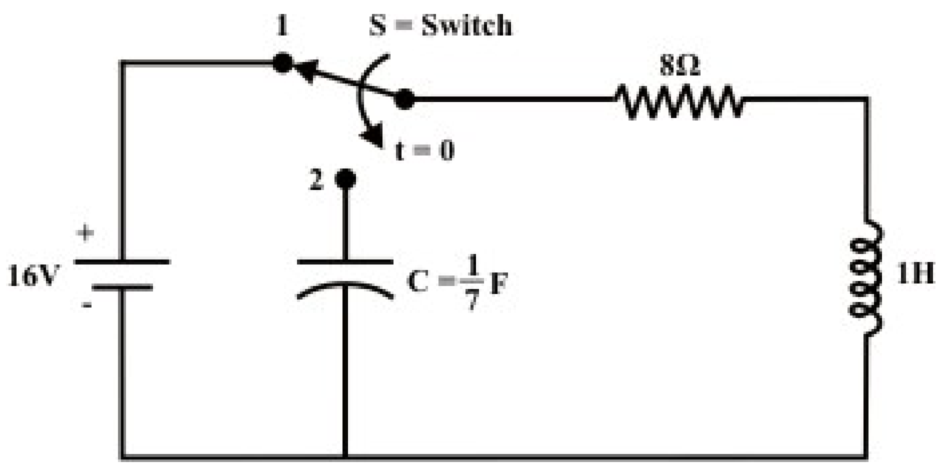

Fig. 11.12

Determine $i(t)$ *for* $t \geq 0$ and sketch its waveform. Remarks on the system's response. [8]

(Ans. $i(t) = \left(\frac{7}{3}e^{-7t} - \frac{1}{3}e^{-t}\right) amps.$)

T.11.10 The switch 'S' in the circuit of Fig.11.13 is opened at $t = 0$ having been closed for a long time.

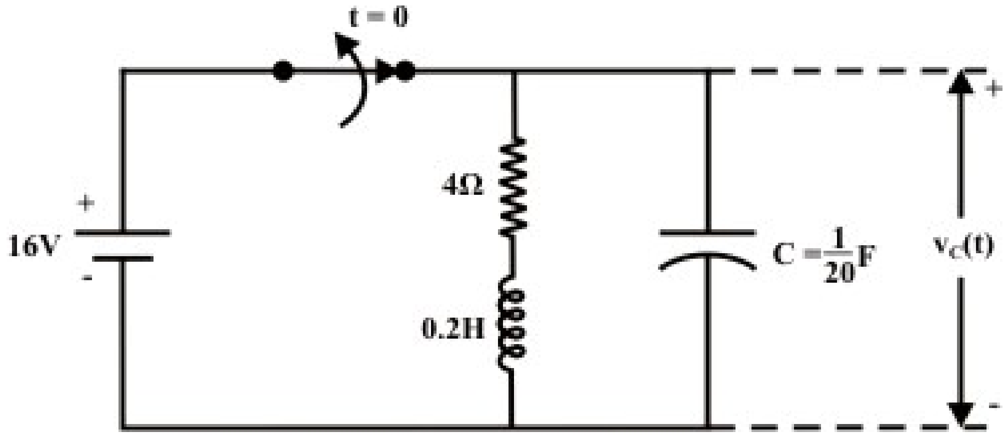

Fig. 11.13

Determine (i) $v_c(t)$ *for* $t \geq 0$ (ii) how long must the switch remain open for the voltage $v_c(t)$ to be less than 10% ot its value at $t = 0$? [10]

(Ans. (i) $(i)\ v_c(t) = (16 + 240t)e^{-10t}$ $(ii)\ 0.705\sec.$)

T.11.11 For the circuit shown in Fig.11.14, find the capacitor voltage $v_c(t)$ and inductor current $i_L(t)$ for all t $(t < 0 \text{ and } t \geq 0)$. [10]

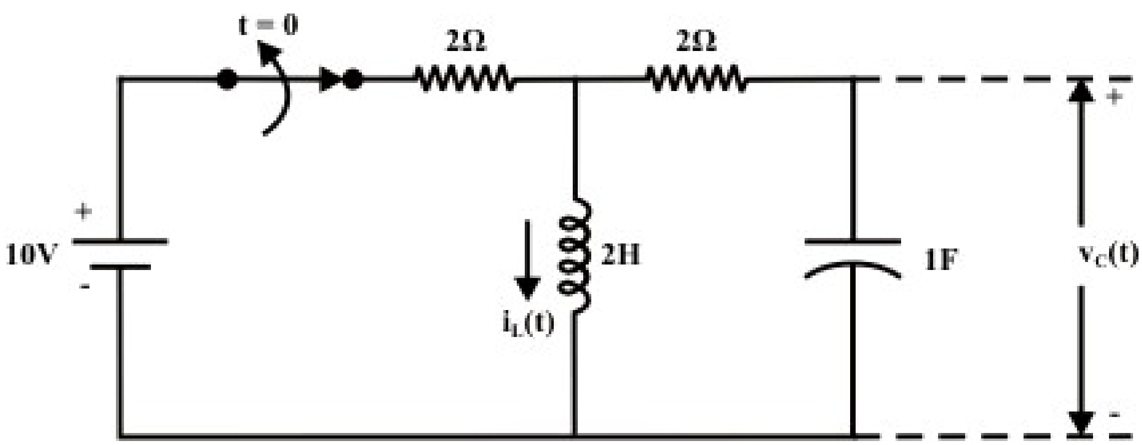

Fig. 11.14

Plot the wave forms $v_c(t)$ and $i_L(t)$ for $t \geq 0$.

(Ans. $v_{c(t)} = 10e^{-0.5t}\sin(0.5t)$; $i_L(t) = 5\left(\cos(0.5t) - \sin(0.5t)\right)e^{-0.5t}$)

T.11.12 For the parallel circuit RLC shown in Fig.11.15, Find the response

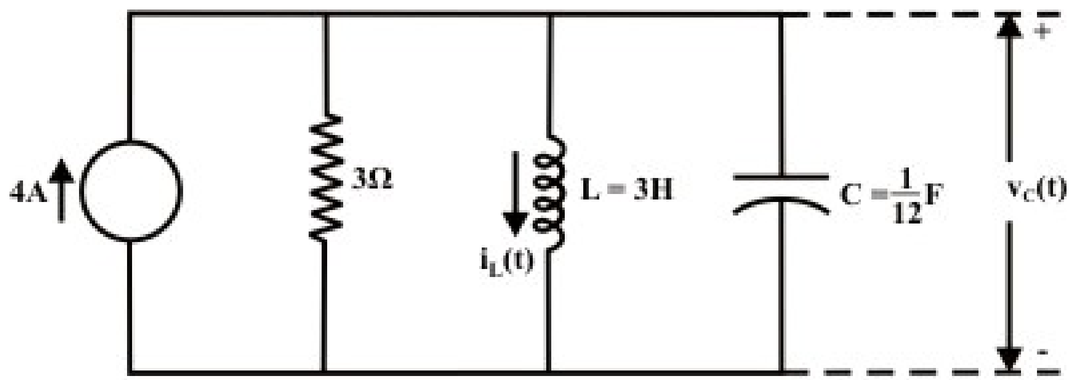

Fig. 11.15

of $i_L(t)$ *and* $v_c(t)$ respectively. [10]

(Ans. $i_L(t) = \left[4 - 4e^{-2t}\left(1 + 2t\right)\right]$ *amps.*; $v_c(t) = 48\,t\,e^{-2t}$ *volt.*)